Indian Statistical Institute
Bangalore Centre
Second Semester Examination 2005-2006
B.Math. (Hons.) III Year

Time: 3 hrs Date: 04-12-2006 Instructor: G Misra

Each question carries 10 marks. The maximum you can score is 40. You may use your class-room notes.

(1) (i) Let f be an entire function with the property that $\Re f$ is bounded, where $\Re f$ is the real part of f. Show that f is constant
(ii) Let f be an entire function with the property that $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that f is a constant function.

(2) (i) Find a conformal map from $\mathbb{D}$ onto itself with $f(0) = 0$ and $f(\frac{1}{2}) = \frac{i}{2}$.
(ii) Let $g : \mathbb{D}_r(a) \to \mathbb{D}_R(g(a))$ be a holomorphic function. Prove that
$$|g(z+a) - g(a)| \leq \frac{R}{r}|z| \text{ for all } z \in \mathbb{D}_r(a).$$

(3) Find a one to one conformal map of the semi-disc $\{z \in \mathbb{C} : \Im z > 0, |z - \frac{1}{2}| < \frac{1}{2}\}$ onto the upper half plane, $\Im z$ is the imaginary part of z.

(4) Let Ω be a connected open set in $\mathbb{C}$ and assume that $\Omega_0 \subseteq \Omega$ has at least one point of accumulation in Ω. Let $\{f_n\}_{n\geq 1}$ be a sequence of functions holomorphic on Ω which is uniformly bounded on any compact subset of Ω. Suppose that $\{f_n(\omega)\}_{n\geq 1}$ converges for all $\omega \in \Omega_0$ as $n \to \infty$. Then $\{f_n\}_{n\geq 1}$ converges uniformly on compact subsets of Ω.

(Hint: Recall that if $\mathcal{F}$ is a family of holomorphic functions on Ω with the property that for each compact subset $K \subseteq \Omega$, there exists $M_K > 0$ such that $|f(z)| \leq M_K$ for all $z \in K$ and for all $f \in \mathcal{F}$ then $\mathcal{F}$ is a normal family.)

(5) Let $f : \mathbb{C} \to \mathbb{C}_{0,1}$ be analytic and ϱ be a metric on $\mathbb{C}_{0,1}$ of curvature atmost -1, where $\mathbb{C}_{0,1}$ is the set of all complex number minus $\{1, 0\}$.
(a) Show that $f^*(\varrho) \leq \lambda_R$ on any disc $\mathbb{D}_R$, where $\lambda_R(z) = \frac{2R}{R^2 - |z|^2}$ is the Poincare metric of the disc $\mathbb{D}_R$.
(b) Show that $f^*(\varrho) = 0$.
(c) Derive Picard's little theorem.